Inverse subsemigroups of finite index in finitely generated inverse semigroups

Citation for published version:
Alali, AS & Gilbert, N 2017, 'Inverse subsemigroups of finite index in finitely generated inverse semigroups', *Semigroup Forum*, pp. 1-17. https://doi.org/10.1007/s00233-017-9886-1

Digital Object Identifier (DOI):
10.1007/s00233-017-9886-1

Published In:
Semigroup Forum

INVERSE SUBSEMIGROUPS OF FINITE INDEX IN FINITELY GENERATED INVERSE SEMIGROUPS

AMAL ALALI AND N.D. GILBERT

Abstract. We study some aspects of Schein's theory of cosets for closed inverse subsemigroups of inverse semigroups. We establish an index formula for chains of subsemigroups, and an analogue of M. Hall's Theorem on the number of cosets of a fixed finite index. We then investigate the relationships between the following properties of a closed inverse submonoid of an inverse monoid: having finite index; being a recognizable subset; being a rational subset; being finitely generated (as a closed inverse submonoid). A remarkable result of Margolis and Meakin shows that these properties are equivalent for a closed inverse submonoid of a free inverse monoid.

1. Introduction

A generalisation of the concept of *coset*, from groups to inverse semigroups, was proposed by Schein in [17]. There are three essential ingredients to this generalisation: firstly, cosets are only to be defined for an inverse subsemigroup L of an inverse semigroup S that is *closed* in the natural partial order on S; secondly, an element $s \in S$ will only determine a coset if $ss^{-1} \in L$; and thirdly, the coset is finally obtained by taking the closure (again with respect to the natural partial order on S) of the subset Ls. The details of this construction are presented in Section 2. In fact, Schein takes as his starting point a characterization of cosets in groups due to Baer [3] (see also [5]): a subset C of a group G is a coset of some subgroup H of G if and only if C is closed under the ternary operation $(a, b, c) \mapsto ab^{-1}c$ on G.

A closed inverse subsemigroup L of an inverse semigroup S has *finite index* if and only if there are only finitely many distinct cosets of L in S. In contrast to the situation in group theory, finite index can arise because of the relative paucity of possible coset representatives satisfying $ss^{-1} \in L$. For example, in the free inverse monoid $FIM(a, b)$, the inverse subsemigroup $FIM(a)$ has finite index. This fact is a consequence of a remarkable theorem of Margolis and Meakin, characterising the closed inverse submonoids in a free inverse monoid $FIM(X)$ with X finite:

Theorem 1.1. [10, Theorem 3.7] *Let X be a finite set, and let L be a closed inverse submonoid of the free inverse monoid $FIM(X)$. Then the following conditions are equivalent:*

(a) *L is recognised by a finite inverse automaton,*
(b) *L has finite index in $FIM(X)$,*
(c) *L is a recognisable subset of $FIM(X)$,*
(d) *L is a rational subset of $FIM(X)$,*
(e) *L is finitely generated as a closed inverse submonoid of $FIM(X)$.*

2010 *Mathematics Subject Classification.* Primary 20M18; Secondary 20M30, 20M10.
Key words and phrases. inverse semigroup, coset, index.

Condition (e) of Theorem 1.1 asserts the existence of a finite set $Y \subset FIM(X)$ such that L is equal to the closed inverse submonoid generated by Y. The original statement of the theorem in [10] includes an extra condition related to immersions of finite graphs, which we have omitted.

Our aims in the present paper are to present some basic facts about closed inverse subsemigroups of finite index, and to study the relationships between the conditions given in Theorem 1.1 when $FIM(X)$ is replaced by an arbitrary inverse monoid.

In Section 2 we give an introduction to the concept of cosets in inverse semigroups, and the action of an inverse semigroup on a set of cosets, based closely on the ideas of Schein [16, 17]. We establish an *index formula*, relating the indices $[S : K], [S : H]$ and $[H : K]$ for closed inverse subsemigroups H, K of an inverse semigroup S with $E(S) \subseteq K \subseteq H$ in Theorem 2.8, and an analogue of M. Hall's Theorem for groups, that in a free group of finite rank, there are only finitely many subgroups of a fixed finite index, in Theorem 2.14.

Our work based on Theorem 1.1 occupies section 3, and is summarised in Theorem 4.1. We show that, in an arbitrary finitely generated inverse monoid M, a closed inverse submonoid has finite index if and only if it is recognisable, in which case it is rational and finitely generated as a closed inverse submonoid, but that finite generation (as a closed inverse submonoid) is a strictly weaker property. This is not a surprise, since any inverse semigroup with a zero is finitely generated in the closed sense.

The authors thank Mark Lawson and Rick Thomas for very helpful comments, and in particular for their shrewd scrutiny of Lemma 3.10.

2. Cosets of closed inverse subsemigroups

Let S be an inverse semigroup with semilattice of idempotents $E(S)$. Recall that the *natural partial order* on S is defined by

$$s \leqslant t \iff \text{there exists } e \in E(S) \text{ such that } s = et\,.$$

A subset $A \subseteq S$ is *closed* if, whenever $a \in A$ and $a \leqslant s$, then $s \in A$. The closure $B^{\uparrow}$ of a subset $B \subseteq S$ is defined as

$$B^{\uparrow} = \{s \in S : s \geqslant b \text{ for some } b \in B\}\,.$$

A subset L of S is *full* if $E(S) \subseteq L$.

An *atlas* in S is a subset $A \subseteq S$ such that $AA^{-1}A \subseteq A$: that is, A is closed under the *heap* ternary operation $\langle a, b, c\rangle = ab^{-1}c$ (see [3]). Since, for all $a \in A$ we have $\langle a, a, a\rangle = a$, we see that A is an atlas if and only if $AA^{-1}A = A$. A *coset* C in S is a closed atlas: that is, C is both upwards closed in the natural partial order on S and is closed under the heap operation $\langle \cdots \rangle$.

Let X be a set and $\mathscr{I}(X)$ its symmetric inverse monoid. Let $\rho : S \to \mathscr{I}(X)$ be a faithful representation of S on X, and write $x(s\rho)$ as $x \lhd s$.

The principal characterisations of cosets that we need are due to Schein:

Theorem 2.1. [17, Theorem 3.] *Let C be a non-empty subset of an inverse semigroup S. Then the following are equivalent:*

(a) *C is a coset,*
(b) *there exists a closed inverse subsemigroup L of S such that, for all $s \in C$, we have $ss^{-1} \in L$ and $C = (Ls)^{\uparrow}$.*

(c) *there exists a closed inverse subsemigroup K of S such that, for all $s \in C$, we have $s^{-1}s \in K$ and $C = (sK)^{\uparrow}$.*

Proof. (a) $\Longrightarrow$ (b): Let $Q = CC^{-1} = \{ab^{-1} : a, b \in C\}$. Then Q is an inverse subsemigroup of S, since, for all $a, b, c, d \in C$ we have

- $(ab^{-1})(cd^{-1}) = (ab^{-1}c)d^{-1} = \langle a, b, c\rangle d^{-1} \in Q$,
- $(ab^{-1})^{-1} = ba^{-1} \in Q$.

Set $L = Q^{\uparrow}$: then L is a closed inverse subsemigroup of S. Let $s \in C$. Obviously $ss^{-1} \in Q \subseteq L$. Moreover, given any $c \in C$ we have $c \geqslant c(s^{-1}s) = (cs^{-1})s \in Qs \subseteq Ls$, so that $C \subseteq (Ls)^{\uparrow}$. Conversely, if $x \in (Ls)^{\uparrow}$, we have $x \geqslant us$ for some $u \in L$, with $u \geqslant ab^{-1}$ for some $a, b \in C$. Hence $x \geqslant us \geqslant ab^{-1}c = \langle a, b, c\rangle \in C$. Since C is closed, $x \in C$ and therefore $(Ls)^{\uparrow} \subseteq C$.

(b) $\Longrightarrow$ (a): The subset $(Ls)^{\uparrow}$ is a coset, since it is closed by definition, and if $h_i \in (Ls)^{\uparrow}$ we have $h_i \geqslant t_i s$ for some $t_i \in L$. Then $\langle h_1, h_2, h_3\rangle = h_1 h_2^{-1} h_3 \geqslant t_1 ss^{-1} t_2^{-1} t_3 s \in Ls$ since $ss^{-1} \in L$. It follows that $(Ls)^{\uparrow}$ is closed under the heap operation $\langle \cdots \rangle$.

For (a) $\Longleftrightarrow$ (c): we proceed in the same way, with $K = (C^{-1}C)^{\uparrow}$. □

For the rest of this paper, all cosets will be *right* cosets, of the form $(Ls)^{\uparrow}$.

Proposition 2.2. [17, Proposition 5.] *A coset C that contains an idempotent $e \in E(S)$ is an inverse subsemigroup of S, and in this case $C = (CC^{-1})^{\uparrow}$.*

Proof. If $a, b \in C$ then $ab \geqslant aeb = \langle a, e, b\rangle \in C$ and since C is closed, we have $ab \in C$. Furthermore, $a^{-1} \geqslant ea^{-1}e = \langle e, a, e\rangle \in C$ and so $a^{-1} \in C$. Hence C is an inverse subsemigroup.

Now $ab^{-1} \in CC^{-1}$ and $ab^{-1} \geqslant ab^{-1}e = \langle a, b, e\rangle \in C$. Since C is closed we have $(CC^{-1})^{\uparrow} \subseteq C$. But if $x \in C$ then $x \geqslant xe \in CC^{-1}$ and so $x \in (CC^{-1})^{\uparrow}$. Therefore $C = (CC^{-1})^{\uparrow}$. □

Now if L is a closed inverse subsemigroup of S, a coset of L is a subset of the form $(Ls)^{\uparrow}$ where $ss^{-1} \in L$. Suppose that C is such a coset: then Theorem 2.1 associates to C the closed inverse subsemigroup $(CC^{-1})^{\uparrow}$.

Proposition 2.3. [17, Proposition 6.] *Let L be a closed inverse subsemigroup of S.*

(a) *Suppose that C is a coset of L. Then $(CC^{-1})^{\uparrow} = L$.*
(b) *If $t \in C$ then $tt^{-1} \in L$ and $C = (Lt)^{\uparrow}$. Hence two cosets of L are either disjoint or they coincide.*
(c) *Two elements $a, b \in S$ belong to the same coset C of L if and only if $ab^{-1} \in L$.*

Proof. (a) If $c_i \in C$ $(i = 1, 2)$ then there exists $l_i \in L$ such that $c_i \geqslant l_i s$. Hence $c_1 c_2^{-1} \geqslant l_1 ss^{-1} l_2^{-1} \in L$, and so $CC^{-1} \subseteq L$. Since L is closed, $(CC^{-1})^{\uparrow} \subseteq L$. On the other hand, for any $l \in L$ we have $l = ll^{-1}l \geqslant lss^{-1}l^{-1}l = (ls)(l^{-1}ls)^{-1} \in CC^{-1}$ and so $L \subseteq (CC^{-1})^{\uparrow}$.

(b) If $C = (Ls)^{\uparrow}$ and $t \in C$ then, for some $l \in L$ we have $t \geqslant ls$. Then $tt^{-1} \geqslant lss^{-1}l^{-1} \in L$, and since L is closed, $tt^{-1} \in L$. Moreover, if $u \in (Lt)^{\uparrow}$ then for some $k \in L$ we have $u \geqslant kt \geqslant kls$ and so $u \in (Ls)^{\uparrow}$. Hence if $t \in (Ls)^{\uparrow}$ then $(Lt)^{\uparrow} \subseteq (Ls)^{\uparrow}$. Now $ls = (ls)(ls)^{-1}t = lss^{-1}l^{-1}t$ and so $l^{-1}ls = l^{-1}lss^{-1}l^{-1}t = ss^{-1}l^{-1}t \in Lt$. Since $s \geqslant l^{-1}ls$, we deduce that $s \in (Lt)^{\uparrow}$. Hence $(Ls)^{\uparrow} \subseteq (Lt)^{\uparrow}$.

(c) Suppose that $a, b \in (Ls)^{\uparrow}$. Then for some $k, l \in L$ we have $a \geqslant ks$ and $b \geqslant ls$: hence $ab^{-1} \geqslant kss^{-1}l^{-1} \in L$ and so $ab^{-1} \in L$. On the other hand, suppose that $ab^{-1} \in L$. Then $aa^{-1} \geqslant a(b^{-1}b)a^{-1} = (ab^{-1})(ab^{-1})^{-1} \in L$, and similarly $bb^{-1} \in L$. We note that $a = (aa^{-1})a \in La$ and similarly $b \in Lb$. Then $a \geqslant a(b^{-1}b) = (ab^{-1})b$ and so $a \in (Lb)^{\uparrow}$. As in part (b) we deduce that $(La)^{\uparrow} \subset (Lb)^{\uparrow}$. By symmetry $(La)^{\uparrow} = (Lb)^{\uparrow}$ and this coset contains a and b. □

Example 2.4. Let E be the semilattice of idempotents of S. The property that E is closed is exactly the property that S is *E–unitary*. In this case, for any $s \in S$, we have

$$\begin{aligned}(Es)^{\uparrow} &= \{t \in S : t \geqslant es \text{ for some } e \in E\} \\ &= \{t \in S : t \geqslant u \leqslant s \text{ for some } u \in S\} \\ &= \{t \in S : s, t \text{ have a lower bound in } S\}.\end{aligned}$$

We see that $(Es)^{\uparrow}$ is precisely the σ–class of s, where σ is the minimum group congruence on S, see [9, Section 2.4]. Hence every element $t \in S$ lies in a coset of E, and the set of cosets is in one-to-one correspondence with the maximum group image $\widehat{S}$ of S.

Remark 2.5. Let L be a closed inverse subsemigroup of an inverse semigroup S. Then the union U, of all the cosets of L is a subset of S but need not be all of S, and is not always a subsemigroup of S.

We illustrate this remark in the following example.

Example 2.6. Fix a set X and recall that the *Brandt semigroup* B_X is defined as follows. As a set, we have

$$B_X = \{(x, y) : x, y \in X\} \cup \{0\}$$

with

$$(u, v)(x, y) = \begin{cases} (u, y) & \text{if } v = x \\ 0 & \text{if } v \neq x \end{cases}$$

and $0(x, y) = 0 = (x, y)0$. The idempotents of B_X are the elements (x, x) for $x \in X$ and 0. Hence $0 \leqslant (x, y)$ for all $x, y \in X$ and $(u, v) \leqslant (x, y)$ if and only if $(u, v) = (x, y)$. If a closed inverse semigroup L of B_X contains (x, y) with $x \neq y$ then $(x, y)(x, y) = 0 \in L$ and so $L = B_X$. Therefore the only proper closed inverse subsemigroups are the subsemigroups $E_x = \{(x, x)\}$ for $x \in X$. An element $(x, y) \in B_X$ then determines the coset

$$(E_x(x, y))^{\uparrow} = \{(x, y)\} .$$

Hence there are $|X|$ distinct cosets of E_x and their union is

$$U = \{(x, y) : y \in X\} .$$

Proposition 2.7.

(a) *Let L be a closed inverse subsemigroup of an inverse semigroup S and let U be the union of all the cosets of L in S. Then $U = \{s \in S : ss^{-1} \in L\}$ and therefore $U = S$ if and only if L is full.*

(b) *U is a closed inverse subsemigroup of S if and only if whenever $e \in E(L)$ and $s \in U$ then $ses^{-1} \in U$ and if, whenever $s \in S$ with $ss^{-1} \in L$, then $s^{-1}s \in L$.*

Proof. (a) The coset $(Lu)^{\uparrow}$ containing $u \in S$ exists if and only if $uu^{-1} \in L$.

(b) Suppose that L satisfies the given conditions. If $s, t \in U$ then $ss^{-1}, tt^{-1} \in L$ and
$$(st)(st)^{-1} = s(tt^{-1})s^{-1} \in U$$
which implies that $st \in U$, and $s^{-1}s \in L$ which implies that $s^{-1} \in U$. Hence U is an inverse subsemigroup. Conversely, if U is an inverse subsemigroup and $s \in U$, then $s^{-1} \in U$ which implies that $s^{-1}s \in L$, and if $s \in U$ and $e \in E(L)$ then $e \in U$ and so $se \in U$ which implies that $(se)(se)^{-1} = ses^{-1} \in U$. $\square$

2.1. **The index formula.** The *index* of the closed inverse subsemigroup L in an inverse semigroup S is the cardinality of the set of right cosets of L, and is written $[S : L]$. Note that the mapping $(Ls)^{\uparrow} \to (s^{-1}L)^{\uparrow}$ is a bijection from the set of right cosets to the set of left cosets. A *transversal* to L in S is a choice of one element from each right coset of L. For a transversal $\mathcal{T}$, we have the union
$$U = \bigcup_{t \in \mathcal{T}} (Lt)^{\uparrow},$$
as in Proposition 2.7, and each element $u \in U$ satisfies $u \geqslant ht$ for some $h \in L$, $t \in \mathcal{T}$.

Theorem 2.8. *Let S be an inverse semigroup and let H and K be two closed inverse subsemigroups of S with $K \subseteq H$ and K full in S. Suppose that K has finite index in H and H has finite index in S. Then K has finite index in S and*
$$[S : K] = [S : H][H : K].$$

Proof. Since K is full in S, then so is H and for transversals $\mathcal{T}, \mathcal{U}$ we have
$$S = \bigcup_{t \in \mathcal{T}} (Ht)^{\uparrow} \quad \text{and} \quad H = \bigcup_{u \in \mathcal{U}} (Ku)^{\uparrow}.$$
Therefore
$$S = \{s \in S : s \geqslant ht \quad \textit{for some}\; t \in \mathcal{T}, h \in H\},$$
and
$$H = \{s \in S : s \geqslant ku \quad \textit{for some}\; u \in \mathcal{U}, k \in K\}.$$
Now if $s \geqslant ht$ and $h \geqslant ku$ then $s \geqslant kut$. Then $s \in (Kut)^{\uparrow}$ and $(Kut)^{\uparrow}$ is a coset of K in S, since K is full in S and therefore $(ut)\,(ut)^{-1} \in K$.
Hence
$$S = \bigcup_{\substack{u \in U \\ t \in \mathcal{T}}} (Kut)^{\uparrow}.$$

It remains to show that all the cosets $(Kut)^{\uparrow}$ are distinct. If $(Ku't')^{\uparrow} = (Kut)^{\uparrow}$ then by part (c) of Proposition 2.3, $u't't^{-1}u^{-1} \in K$ and so $u't't^{-1}u^{-1} \in H$. Now $t't^{-1} \geqslant (u')^{-1}u't't^{-1}u^{-1}u \in H$ since $u, u' \in H$, and so $t't^{-1} \in H$ since H is closed. This implies that $(Ht')^{\uparrow} = (Ht)^{\uparrow}$ and it follows that $t' = t$.

Now $u't't^{-1}u^{-1} \in K$. Since $t' = t$, then $t't^{-1} \in E(S)$ and so $u'u^{-1} \geqslant u't't^{-1}u^{-1}$. But K is closed, so $u'u^{-1} \in K$ and $(Ku')^{\uparrow} = (Ku)^{\uparrow}$. Hence $u' = u$. Consequently, all the cosets $(Kut)^{\uparrow}$ are distinct. $\square$

Recall from Example 2.4 that the property that $E(S)$ is closed is expressed by saying that S is E–unitary and that in this case, the set of cosets of $E(S)$ is in one-to-one correspondence with the maximum group image $\widehat{S}$ of S.

Proposition 2.9. *Let S be an E–unitary inverse semigroup. Then:*

(a) *$E(S)$ has finite index if and only if the maximal group image $\widehat{S}$ is finite, and $[S : E] = |\widehat{S}|$,*
(b) *if $E(S)$ has finite index in S then, for any closed, full, inverse subsemigroup L of S we have*

$$[S : L] = |\widehat{S}| \,/\, |\widehat{L}|$$

Proof. Part (a) follows from our previous discussion. For part (b) we have $E(S) \subseteq L \subseteq S$ and so if the index $[S : E]$ is finite then so are $[S : L]$ and $[L : E]$ with $[S : E] = [S : L][L : E]$. But now $[S : E] = |\widehat{S}|$ and $[L : E] = |\widehat{L}|$. □

The index formula in Theorem 2.8 can still be valid when K is not full in S as we show in the following Example.

Example 2.10. We work in the symmetric inverse monoid $\mathscr{I}_3 = \mathscr{I}(\{1, 2, 3\})$, and take $L = stab(1) = \{\sigma \in \mathscr{I}(X) : 1\sigma = 1\}$, which is a closed inverse subsemigroup of $\mathscr{I}(X)$ with 7 elements. There are 3 cosets of L in $\mathscr{I}_3$, namely

$$C_1 = \{\sigma \in \mathscr{I}_3 : 1\sigma = 1\} = L,$$
$$C_2 = \{\sigma \in \mathscr{I}_3 : 1\sigma = 2\},$$
$$C_3 = \{\sigma \in \mathscr{I}_3 : 1\sigma = 3\},$$

and so $[\mathscr{I}_3 : L] = 3$.

Now take

$$K = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}.$$

Then K is a closed inverse subsemigroup of L. The domain of each σ in K is $\{1, 2, 3\}$. and so the only coset representatives for K in L are the permutations

$$\mathrm{id} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

But these are elements of K and so $K^{\uparrow} = (K\sigma)^{\uparrow} = K$ and there is just one coset. Hence $[L : K] = 1$.

Now, we calculate the cosets of K in $\mathscr{I}_3$. Each permutation in $\mathscr{I}_3$ is a possible coset representative for K and these produce three distinct cosets by Proposition 2.3(3). Hence $[\mathscr{I}_3 : K] = 3$ and in this example

$$[\mathscr{I}_3 : K] = [\mathscr{I}_3 : L][L : K].$$

The index of a closed inverse subsemigroup L of an inverse semigroup S depends on the availability of coset representatives to make cosets, and so on the idempotents of S contained in L. In particular, we can have $K \subset L$ but $[S : K| < [S : L]$ as the following Example illustrates.

Example 2.11. Consider the free inverse monoid $FIM(x, y)$ and the closed inverse submonoids $K = \langle x^2 \rangle^{\uparrow}$ and $H = \langle x^2, y^2 \rangle^{\uparrow}$. As in [10, 11], we represent elements of $FIM(x, y)$ by Munn trees (P, w) in the Cayley graph $\Gamma(F(x, y), \{x, y\})$ where $F(x, y)$ is the free group on x, y.

Consider a coset $K(P, w)^{\uparrow}$. For this to exist, the idempotent $(P, 1) = (P, w)(P, w)^{-1}$ must be in K and so as a subtree of Γ, P can only involve vertices in $F(x)$ and edges between them. Since $w \in P$ we must have $w \in F(x)$. It is then easy to see that there are only two cosets, K and $(Kx)^{\uparrow}$ and so $[FIM(x, y) : K] = 2$. Similarly, $[H : K] = 1$.

Now consider a coset $H(P,w)^{\uparrow}$. Now $(P,1) \in H$ and so P must be contained in the subtree of Γ spanned by the vertices of the subgroup $\langle x^2, y^2 \rangle \subset F(x,y)$, and $w \in P$. If $w \in \langle x^2, y^2 \rangle$ then $(P,w) \in H$ and $H(P,w)^{\uparrow} = H$. Otherwise, $w = ux$ or $w = uy$ with $u \in \langle x^2, y^2 \rangle$ and it follows that there are three cosets of H in $FIM(x,y)$, namely $H, (Hx)^{\uparrow}$ and $(Hy)^{\uparrow}$. Therefore

$$[FIM(x,y) : H] = 3 > [FIM(x,y) : K] = 2 .$$

These calculations also follow from results of Margolis and Meakin, see [10, Lemma 3.2].

We note that the index formula fails to hold. This does not contradict Theorem 2.8, since K is not full in $FIM(x,y)$.

2.2. **Marshall Hall's Theorem for inverse semigroups.** Our next aim is to derive an analogue of Marshall Hall's Theorem (see [7] and [2]) that, in a free group of finite rank, there are only finitely many subgroups of a fixed finite index. We first record some preliminary results on actions on cosets: these results are due to Schein [16] and are presented in [8, Section 5.8]. Let L be a closed inverse subsemigroup of S, and let $s \in S$ with $ss^{-1} \in L$ with $C = (Ls)^{\uparrow}$ Now suppose that $u \in S$ and that $(Cuu^{-1})^{\uparrow} = C$. Then we define $C \triangleleft u = (Cu)^{\uparrow}$.

Lemma 2.12. *The condition $(Cuu^{-1})^{\uparrow} = C$ for $C \triangleleft u$ to be defined is equivalent to the condition that $suu^{-1}s^{-1} \in L$.*

It follows from Lemma 2.12 that the condition $suu^{-1}s^{-1} \in L$ does not depend on the choice of coset representative s. This is easy to see directly. If $(Ls)^{\uparrow} = (Lt)^{\uparrow}$ then, by part (c) of Proposition 2.3 we have $st^{-1} \in L$. Then

$$tuu^{-1}t^{-1} \geqslant ts^{-1}suu^{-1}s^{-1}st^{-1} = (st^{-1})^{-1}(suu^{-1}s^{-1})(st^{-1}) \in L$$

and since L is closed, $tuu^{-1}t^{-1} \in L$.

Proposition 2.13. *If $u \in S$ and $(Cuu^{-1})^{\uparrow} = C$ then $(Cu)^{\uparrow} = (Lsu)^{\uparrow}$ and the rule $C \triangleleft u = (Cu)^{\uparrow}$ defines a transitive action of S by partial bijections on the cosets of L.*

We can now prove our version of Marshall Hall's Theorem.

Theorem 2.14. *In a finitely generated inverse semigroup S there are at most finitely many distinct closed inverse subsemigroups of a fixed finite index d.*

Proof. Suppose that the inverse semigroup S is finitely generated and that the closed inverse subsemigroup L of S has exactly d cosets. We aim to construct an inverse semigroup homomorphism

$$\phi_L : S \longrightarrow \mathscr{I}(D),$$

where $\mathscr{I}(D)$ is the symmetric inverse monoid on $D = \{1, ..., d\}$.

Write the distinct cosets of L as $(Lc_1)^{\uparrow}, (Lc_2)^{\uparrow}, \dots, (Lc_d)^{\uparrow}$, with $c_1, c_2, ..., c_d \in S$, and with $(Lc_1)^{\uparrow} = L$. Now take $u \in S$. If $c_j\, u\, u^{-1}\, c_j{}^{-1} \in L$, where $j \in \{1, ..., d\}$, then we can define an action of the element $u \in S$ on the coset $(Lc_j)^{\uparrow}$ of L as follows:

$$(Lc_j)^{\uparrow} \triangleleft u = (Lc_j u)^{\uparrow}.$$

By Proposition 2.13, $(Lc_j u)^{\uparrow}$ is indeed a coset of L, and so $(Lc_j u)^{\uparrow} = (Lc_k)^{\uparrow}$, where $k \in \{1, ..., d\}$. Then we can write $(Lc_j)^{\uparrow} \triangleleft u = (Lc_k)^{\uparrow}$, and this action of u induces an action $j \lhd u = k$ of u on D, and so we get a homomorphism

$$\phi_L : S \longrightarrow \mathscr{I}(D).$$

We now claim that different choices of L give us different homomorphisms ϕ_L, or equivalently, that if $\phi_L = \phi_K$ then $L = K$.

By Proposition 2.2, if $x \in L$ then $L = (Lxx^{-1})^{\uparrow}$. By Lemma 2.12 $L \lhd x$ is defined and is equal to $(Lx)^{\uparrow} = L$. Now suppose that $L \lhd y$ is defined and that $(Ly)^{\uparrow}$ is equal to L. By Lemma 2.12 we have $(Lyy^{-1})^{\uparrow} = L$. Hence $yy^{-1} \in L$, and $y = yy^{-1}y \in (Ly)^{\uparrow} = L$. It follows that $\mathrm{stab}(L) = L$ and in the induced action of S on D we have $\mathrm{stab}(1) = L$, so that L is determined by ϕ_L . Therefore, the number of closed inverse subsemigroups of index d is at most the number of homomorphisms $\phi : S \longrightarrow \mathscr{I}(D)$, and since S is finitely generated, this number is finite. $\square$

3. Finite generation and finite index

In this section, we shall look at the properties of closed inverse submonoids of free inverse monoids considered in Theorem 1.1, and the relationships between these properties when we replace a free inverse monoid by an arbitrary inverse monoid. Throughout this section, M will be an inverse monoid generated by a finite subset X. This means that the smallest inverse submonoid $\langle X \rangle$ of M that contains X is M itself: equivalently, each element of M can be written as a product of elements of X and inverses of elements of X, so if we set $A = X \cup X^{-1}$ then each element of M can be written as a product of elements in A. A closed inverse submonoid L of M is said to be *finitely generated as a closed inverse submonoid* if there exists a finite subset $Y \subseteq L$ such that, for each $\ell \in L$ there exists a product w of elements of Y and their inverses such that $\ell \geqslant w$. Equivalently, the smallest closed inverse submonoid of M that contains Y is L. We remark that in [10] the notation $\langle X \rangle$ is used for the smallest *closed* inverse submonoid of M that contains X. We shall use $\langle X \rangle^{\uparrow}$ for this.

We will need to use some ideas from the theory of finite automata and for background information on this topic we refer to [14, 18].

A *deterministic finite state automaton* $\mathcal{A}$ (or just an *automaton* in this section) consists of

- a finite set S of *states*,
- a finite *input alphabet* A,
- an *initial state* $s_0 \in S$,
- a partially defined *transition function* $\tau : S \times A \to S$,
- a subset $T \subseteq S$ of *final* states.

We shall write $s \lhd a$ for $\tau(s, a)$ if $\tau(s, a)$ is defined. Given a word $w = a_1 a_2 \cdots a_m \in A^*$ we write $s \lhd w$ for the state $(\dots(s \lhd a_1) \lhd a_2) \lhd \cdots) \lhd a_m$, that is, for the state obtained from s by computing the succesive outcomes, if all are defined, of the transition function determined by the letters of w, with the empty word ε acting by $s \lhd \varepsilon = s$ for all $s \in S$. We normally think of an automaton in terms of its *transition diagram*, in which the states are the vertices of a directed graph and the edge set is a subset of $S \times A$, with an edge (s, a) having source s and target $s \lhd a$.

Let X be a finite set, X^{-1} a disjoint set of formal inverses of elements of X, and $A = X \cup X^{-1}$ An automaton $\mathcal{A}$ with input alphabet A is called a *dual automaton* if, whenever $s \lhd a = t$ then $t \lhd a^{-1} = s$. A dual automaton is called an *inverse* *automaton* if, for each $a \in A$ the partial function $\tau(-,a) : S \to S$ is injective. (See [9, Section 2.1].)

A word $w \in A^*$ is *accepted* or *recognized* by $\mathcal{A}$ if $s_0 \lhd w$ is defined and $s_0 \lhd w \in T$. The set of all words recognized by $\mathcal{A}$ is the *language* of $\mathcal{A}$:

$$\mathscr{L}(\mathcal{A}) = \{w \in A^* : s_0 \lhd w \in T\}\,.$$

A language $\mathscr{L}$ is *recognizable* if it is the language recognized by some automaton. The connection between automata and closed inverse subsemigroups of finite index is made, as in [10], by the coset automaton.

Let M be a finitely generated inverse monoid, generated by $X \subseteq M$, and let L be a closed inverse submonoid of M of finite index. Since M is generated by X, there is a natural monoid homomorphism $\theta : A^* \to M$. The *coset automaton* $\mathcal{C} = \mathcal{C}(M : L)$ is defined as follows:

- the set of states is the set of cosets of L in M,
- the input alphabet is $A = X \cup X^{-1}$,
- the initial state is the coset L,
- the transition function is defined by $\tau((Lt)^{\uparrow}, a) = (Lt(a\theta))^{\uparrow}$,
- the only final state is L.

By Lemma 2.12 and Proposition 2.13, $(Lt)^{\uparrow} \lhd a$ is defined if and only if $t(a\theta)(a\theta)^{-1}t^{-1} \in L$. The following Lemma occurs as [10, Lemma 3.2] for the case that M is the free inverse monoid $FIM(X)$.

Lemma 3.1. *The coset automaton of L in M is an inverse automaton. The language $\mathscr{L}(\mathcal{C}(M : L))$ that it recognizes is*

$$L\theta^{-1} = \{w \in A^* : w\theta \in L\}$$

and $\mathcal{C}(M : L)$ is the minimal automaton recognizing $L\theta^{-1}$.

Proof. It follows from Proposition 2.13 that $\mathcal{C}(M : L)$ is inverse. Suppose that w is recognized by $\mathcal{C}(M : L)$. Then $(w\theta)(w\theta)^{-1} = (ww^{-1})\theta \in L$ and $L(w\theta)^{\uparrow} = L$. From Proposition 2.3, we deduce that $w\theta \in L$. Conversely, suppose that $w = a_{i_1} \dots a_{i_m} \in A^*$ and that $s = w\theta \in L$. For $1 \leqslant k \leqslant m$, write $p_k = a_{i_1} \dots a_{i_k}$, $q_k = a_{i_{k+1}} \dots a_{i_m}$, so that $w = p_k q_k$, and take $s_k = p_k\theta$, so that $s_1 = a_{i_1}\theta$. Then

$$s_1 s_1^{-1} s = s_1 s_1^{-1} s_1 (q_2\theta) = s_1(q_2\theta) = w\theta = s$$

and so $s_1 s_1^{-1} \geqslant ss^{-1} \in L$. Therefore $s_1 s_1^{-1} \in L$ and $L \lhd a_{i_1} = (Ls_1)^{\uparrow}$ is defined. Now suppose that for some k we have that $L \lhd w_k$ is defined and is equal to $(Ls_k)^{\uparrow}$. Then

$$s_{k+1}\, s_{k+1}^{-1}\, s = s_{k+1}\, s_{k+1}^{-1}\, s_{k+1}(q_{k+1}\theta) = s_{k+1}\,(q_{k+1}\theta) = w\theta = s$$

and so $s_{k+1}\, s_{k+1}^{-1} \geqslant ss^{-1} \in L$ and therefore $s_{k+1}s_{k+1}^{-1} \in L$. But

$$s_{k+1}\, s_{k+1}^{-1} = s_k(a_{i_{k+1}}\theta)\,(a_{i_{k+1}}\theta)^{-1} s_k^{-1} \in L\,,$$

and so by Lemma 2.12, $(Ls_k)^{\uparrow} \lhd a_{i_{k+1}}$ is defined and is equal to $(Ls_k(a_{i_{k+1}}\theta))^{\uparrow} = (Ls_{k+1})^{\uparrow}$. It follows by induction that $L \lhd w$ is defined in $\mathcal{C}(M : L)$ and is equal to $(Ls)^{\uparrow} = L$, and so $w \in L(\mathcal{C}(M : X))$. Now by a result of Reutenauer [15, Lemme 1], a connected inverse automaton with one initial and one final state is minimal. $\square$

The set of *rational* subsets of M is the smallest collection that contains all the finite subsets of M and is closed under finite union, product, and generation of a submonoid. Equivalently, $R \subseteq M$ is a rational subset of M if and only if there exists a recognizable subset $Z \subseteq A^*$ with $Z\theta = R$ (see [14, section IV.1]).

We also recall the notion of *star-height* of a rational set (see [4, Chapter III]). Let M be a monoid. Define a sequence of subsets $\mathrm{Rat}_h(M)$, with *star-height* $h \geqslant 0$, recursively as follows:

$$\mathrm{Rat}_0(M) = \{X \subseteq M |\, X \text{ is finite}\},$$

and $\mathrm{Rat}_{h+1}(M)$ consists of the finite unions of sets of the form $B_1 B_2 \cdots B_m$ where each B_i is either a singleton or $B_i = C_i^*$, for some $C_i \in \mathrm{Rat}_h(M)$. It is well known that $\mathrm{Rat}(M) = \bigcup_{h \geqslant 0} \mathrm{Rat}_h(M)$.

A subset S of M is *recognizable* if there exists a finite monoid N, a monoid homomorphism $\phi : M \to N$, and a subset $P \subseteq N$ such that $S = P\phi^{-1}$. For free monoids A^*, Kleene's Theorem (see for example [14, Theorem 5.3.1]) tells us that the rational and recognizable subsets coincide. For finitely generated monoids, we have the following theorem due to McKnight.

Theorem 3.2. *In a finitely generated monoid M, every recognizable subset is rational.*

If M is generated (as an inverse monoid) by X, then as above we have a monoid homomorphism $\theta : A^* \to M$. We say that a subset S of M is *recognized* by an automaton $\mathcal{A}$ if its full inverse image $S\theta^{-1}$ in A^* is recognized by $\mathcal{A}$. We shall use the Myhill-Nerode Theorem [12, 13] to characterize recognizable languages. Let $K \subseteq A^*$ be a language. Two words $u, v \in A^*$ are *indistinguishable by* K if, for all $z \in A^*$, $uz \in K$ if and only if $vz \in K$. We write $u \simeq_K v$ in this case: it is easy to check that $\simeq_K$ is an equivalence relation (indeed, a right congruence) on A^*. Then we have:

Theorem 3.3 (The Myhill-Nerode Theorem)**.** *A language $\mathscr{L}$ is recognizable if and only if the equivalence relation $\simeq_{\mathscr{L}}$ has finitely many classes.*

3.1. **Finite index implies finite generation.** In this section, we consider a closed inverse submonoid L that has finite index in a finitely generated inverse monoid M. We shall show that L is finitely generated as a closed inverse submonoid. Our proof differs from that given in [10, Theorem 3.7] for the case $M = FIM(X)$: instead we generalize the approach taken for groups in [2, Theorem 3.1.4]. Recall that a transversal to L in M is a choice of one representative element from each coset of L. We always choose the element 1_M from the coset L itself. For $s \in S$ we write $\overline{s}$ for the *representative* of the coset that contains s (if it exists), and note the following observations:

Lemma 3.4. *Let $\mathcal{T}$ be a transversal to L in M and define, for $r \in \mathcal{T}$ and $s \in M$, $\delta(r, s) = rs\,(\overline{rs})^{-1}$. Then for all $s, t \in M$, with $ss^{-1}, stt^{-1}s^{-1} \in L$,*

(a) $(Ls)^{\uparrow} = (L\overline{s})^{\uparrow}$
(b) $\overline{\overline{s}t} = \overline{st}$
(c) $s \geqslant \delta(1_M, s)\,\overline{s}$.

Theorem 3.5. *A closed inverse submonoid of finite index in a finitely generated inverse monoid is finitely generated as a closed inverse submonoid.*

Proof. Let M be an inverse monoid generated by a set X. We set $A = X \cup X^{-1}$: then each $s \in M$ can be expressed as a product $s = a_1 a_2 \cdots a_n$ where $a_i \in A$. Suppose that L is a closed inverse subsemigroup of finite index in M. Let $\mathcal{T}$ be a transversal to L in M. Given $h \in L$, we write $h = x_1 x_2 \cdots x_n$ and consider the prefix $h_i = x_1 x_2 \cdots x_i$ for $1 \leqslant i \leqslant n$. Since

$$h_i h_i^{-1} h h^{-1} = h_i h_i^{-1} h_i x_{i+1} \cdots x_n h^{-1} = h_i x_{i+1} \cdots x_n h^{-1} = hh^{-1},$$

we have $h_i h_i^{-1} \geqslant hh^{-1}$. But $hh^{-1} \in L$ and L is closed, so that $h_i h_i^{-1} \in L$. Therefore the coset $(Lh_i)^{\uparrow}$ exists, and so does the representative $\overline{h_i}$. Now

$$h = x_1 x_2 \cdots x_n \geqslant x_1 \cdot \overline{h_1}^{-1} \overline{h_1} \cdot x_2 \cdot \overline{h_2}^{-1} \overline{h_2} \cdot x_3 \cdot \overline{h_3}^{-1} \cdots \overline{h_{n-1}} \cdot x_n \,.$$

By part (b) of Lemma 3.4 we have $\overline{h_j} = \overline{\overline{h_{j-1}} x_j}$ and so

$$h \geqslant x_1 \cdot \overline{x_1}^{-1} \overline{x_1} \cdot x_2 \cdot (\overline{\overline{h_1} x_2})^{-1} \cdot \overline{h_2} \cdot x_3 \cdot (\overline{\overline{h_2} x_3})^{-1} \cdots \overline{h_{n-1}} \cdot x_n \,.$$

Now using the elements $\delta(r,s)$ from Lemma 3.4, and noting that $1_M = \overline{x_1 x_2 \cdots x_n}$, we have

$$h \geqslant \delta(1_M, x_1)\delta(\overline{x_1}, x_2)\delta(\overline{h_2}, x_3) \cdots \delta(\overline{h_{n-1}}, x_n) \,.$$

Finally, since $(Lrs)^{\uparrow} = (L\overline{rs})^{\uparrow}$ then it follows from Proposition 2.3(3) that $\delta(r,s) \in L$. Hence L is generated as a closed inverse submonoid of M by the elements $\delta(r,x)$ with $r \in \mathcal{T}$ and $x \in A$. □

3.2. **Recognizable closed inverse submonoids.**

Theorem 3.6. *Let L be a closed inverse submonoid of a finitely generated inverse monoid M. Then the following are equivalent:*

(a) *L is recognized by a finite inverse automaton,*
(b) *L has finite index in M,*
(c) *L is a recognizable subset of M.*

Proof. If L has finite index in M, then by Lemma 3.1, its coset automaton $\mathcal{C}(M:L)$ is a finite inverse automaton that recognizes L. Conversely, suppose that $\mathcal{A}$ is a finite inverse automaton that recognizes L. Again by Lemma 3.1, the coset automaton $\mathcal{C}$ is minimal, and so must be finite. Hence (a) and (b) are equivalent.

If (b) holds, then as in the proof of Theorem 2.14, we obtain a homomorphism $M \to \mathscr{I}(D)$ for which L is the inverse image of the stabilizer of 1. Therefore (b) implies (c).

We have M generated by X, with $A = X \cup X^{-1}$, and a monoid homomorphism $\theta : A^* \to M$. To prove that (c) implies (b), suppose that L is recognizable and so the language $\mathscr{L} = \{w \in A^* : w\theta \in L\}$ is recognizable. By Theorem 3.3, the equivalence relation $\simeq_{\mathscr{L}}$ on A^* has finitely many classes. We claim that if $u \simeq_{\mathscr{L}} v$ and if $(L(u\theta))^{\uparrow}$ exists, then $(L(v\theta))^{\uparrow}$ exists and $(L(u\theta))^{\uparrow} = (L(v\theta))^{\uparrow}$. Now $(u\theta)\,(u\theta)^{-1} = (uu^{-1})\theta \in L$ and hence $uu^{-1} \in \mathscr{L}$. But by assumption $u \simeq_{\mathscr{L}} v$, and so $vu^{-1} \in \mathscr{L}$, which implies that $(v\theta)(u\theta)^{-1} \in L$. By part (c) of Proposition 2.3, $(L(v\theta))^{\uparrow}$ exists and $(L(v\theta))^{\uparrow} = (L(u\theta))^{\uparrow}$. Since $\simeq_L$ has only finitely many classes, there are only finitely many cosets of L in M. Hence (c) implies (b). □

3.3. **Rational Generation.** In this section we give an automata-based proof of part of [10, Theorem 3.7]. We adapt the approach used in [6, Theorem II] to the proof of the following theorem of Anisimov and Seifert.

Theorem 3.7. [1, Theorem 3] *A subgroup of a finitely generated group G is a rational subset of G if and only if it is finitely generated.*

Theorem 3.8. *Let L be a closed inverse submonoid of a finitely generated inverse monoid M. Then L is generated as a closed inverse submonoid by a rational subset if and only if L is generated as a closed inverse submonoid by a finite subset.*

Proof. Since finite sets are rational sets, one half of the theorem is trivial.

So suppose that L is generated (as a closed inverse submonoid) by some rational subset Y of L. As above, if M is generated (as an inverse monoid) by X, we take $A = X \cup X^{-1}$, and let θ be the obvious map $A^* \to M$. Then $Z = (Y \cup Y^{-1})^*$ is rational and so there exists a rational language R in A^* such that $R\theta = Z$. The pumping lemma for R then tells us that there exists a constant C such that, if $w \in R$ with $|w| > C$, then $w = uvz$ with $|uv| \leqslant C, |v| \geqslant 1$, and $uv^i z \in R$ for all $i \geqslant 0$. We set

$$U = \{uvu^{-1} : u, v \in A^*, |uv| \leqslant C, (uvu^{-1})\theta \in L\}$$

and $V = \langle U\theta \rangle^{\uparrow}$. Clearly U is finite, and $V \subseteq L$. We claim that $L = V$, and so we shall show that $R\theta \subseteq V$.

We first note that if $w \in R$ and $|w| \leqslant C$ then $w \in U$ (take $u = 1, v = w$) and so $w\theta \in V$. Now suppose that $|w| > C$ but that there exists $n \in L \setminus V$ with $n \geqslant w\theta$. Choose $|w|$ minimal. The pumping lemma gives $w = uvz$ as above. Since $|uz| < |w|$ it follows that $(uz)\theta \in V$.

Moreover,

$$(uvu^{-1})\theta \geqslant (uvzz^{-1}u^{-1})\theta = (uvz)\theta((uz)\theta)^{-1} = (w\theta)\,((uz)\theta)^{-1}.$$

Now $w\theta \in L$ and $(uz)\theta \in V$: since L is closed, $(uvu^{-1})\theta \in L$ and therefore $uvu^{-1} \in U$. Now

$$n \geqslant w\theta = (uvz)\theta \geqslant (uvu^{-1}uz)\theta = (uvu^{-1})\theta\ (uz)\theta \in V.$$

Since V is closed, $n \in V$. But this is a contradiction. Hence $L = V$. □

Corollary 3.9. *If a closed inverse submonoid L of a finitely generated inverse monoid M is a rational subset of M then it is finitely generated as a closed inverse submonoid.*

Proof. If L is a rational subset of M then it is certainly generated by a rational set, namely L itself. □

However, the converse of Corollary 3.9 is not true. We shall use the following Lemma to validate a counterexample in Example 3.12.

Lemma 3.10. *Let M be a semilattice of groups $G_1 \sqcup G_0$ over the semilattice $1 > 0$, and suppose that T is a rational subset of M of star-height h. Then $G_1 \cap T$ is a rational subset of G_1.*

Proof. We proceed by induction on h. If $h = 0$ then T is finite, and $G_1 \cap T$ is a finite subset of G_1 and so is a rational subset of G_1, also of star-height $h_1 = 0$.

If $h > 0$, then, as in section 3, T is a finite union $T = S_1 \cup \cdots \cup S_k$ where each S_j is a product $S_j = R_1 R_2 \cdots R_{m_j}$ and where each R_i is either a singleton subset of M or $R_i = Q_i^*$ for some rational subset Q_i of M of star-height $h-1$ (see [4, Chapter III]). Hence

$$G_1 \cap T = (G_1 \cap S_1) \cup \cdots \cup (G_1 \cap S_k)\,.$$

Consider the subset $G_1 \cap S_j = G_1 \cap R_1R_2 \cdots R_{m_j}$. We claim that

$$G_1 \cap R_1R_2 \cdots R_{m_j} = (G_1 \cap R_1)(G_1 \cap R_2) \cdots (G_1 \cap R_{m_j})\,. \tag{1}$$

The inclusion $\supseteq$ is clear, and so now we suppose that $g \in G_1$ is a product $g = r_1r_2 \cdots r_{m_j}$ with $r_i \in R_i$. If any $r_i \in G_0$ then $g \in G_0$: hence each $r_i \in G_1$ and so $g \in (G_1 \cap R_1)(G_1 \cap R_2) \cdots (G_1 \cap R_{m_j})$, confirming (1).

The factors on the right of (1) are either singleton subsets of G_1, or are of the form $G_1 \cap Q_i^*$ where Q_i is a rational subset of M of star-height $h-1$. However, $G_1 \cap Q_i^* = (G_1 \cap Q_i)^*$: the inclusion $G_1 \cap Q_i^* \supseteq (G_1 \cap Q_i)^*$ is again obvious, and $G_1 \cap Q_i^* \subseteq (G_1 \cap Q_i)^*$ since if $w = x_1 \ldots x_m \in Q_i^*$ and some $x_j \in G_0$ then $w \in G_0$. It follows that if $w \in G_1 \cap Q_i^*$ then $x_j \in G_1$ for all j.

Hence $G_1 \cap T$ is a union of subsets, each of which is a product of singleton subsets of G_1 and subsets of the form $(G_1 \cap Q_i)^*$ where, by induction $G_1 \cap Q_i$ is a rational subset of G_1 of star-height $h_2 \leqslant h-1$. Therefore $G_1 \cap T$ is a rational subset of G_1. □

Corollary 3.11. *Let $L = L_1 \cup L_0$, where $L_j \subseteq G_j$, be an inverse submonoid of M that is also a rational subset of M. Then L_1 is a rational subset of G_1.*

Proof. Take $T = L$: then $G_1 \cap L = L_1$ is a rational subset of G_1. □

Now, we show that the converse of Corollary 3.9 is not true.

Example 3.12. Let F_2 be a free group of rank 2 and consider the semilattice of groups $M = F_2 \sqcup F_2^{ab}$ determined by the abelianisation map $\alpha : F_2 \to F_2^{ab}$. The kernel of α is the commutator subgroup F_2' of F_2 and we let K be the closed inverse submonoid $F_2' \sqcup \{\mathbf{0}\}$.

$$\begin{tikzcd} F_2' \arrow[r] \arrow[d] & F_2 \arrow[d, "\alpha"] \\ \{\mathbf{0}\} \arrow[r] & F_2^{ab} \end{tikzcd}$$

Now K is generated (as a closed inverse submonoid) by $\{\mathbf{0}\}$ and so is finitely generated. But F_2' is not finitely generated as a group (see [2, Example III.4(4)]) and so is not a rational subset of F_2 by Theorem 3.7. Therefore, by Corollary 3.11, K is not a rational subset of M.

This example also gives us a counterexample to the converse of Theorem 3.5: K is finitely generated as a closed inverse submonoid, but has infinite index in M.

4. Conclusion

We summarize our findings about the conditions considered by Margolis and Meakin in [10, Theorem 3.5].

Theorem 4.1. *Let L be a closed inverse submonoid of the finitely generated inverse monoid M and consider the following properties that L might possess:*

(a) *L is recognized by a finite inverse automaton,*
(b) *L has finite index in M,*

(c) *L is a recognizable subset of M,*
(d) *L is a rational subset of M,*
(e) *L is finitely generated as a closed inverse submonoid of M.*

Then properties (a), (b) and (c) are equivalent: each of them implies (d), and (d) implies (e). The latter two implications are not reversible.

Proof. The equivalence of (a), (b) and (c) was established in Theorem 3.6, and that (d) implies (e) in Corollary 3.9. The implication that (c) implies (d) is McKnight's Theorem 3.2.

Counterexamples for (e) implies (d) and (e) implies (b) are given in Example 3.12 □

Remark 4.2. In Theorem 4.1, if M is a group and L a subgroup of M, then by the result [1, Theorem 3] of Anisimov and Seifert (stated here as Theorem 3.7), properties (d) and (e) are equivalent, and each is implied by the equivalent properties (a), (b),(c). This implication remains irreversible, since (for example) finitely generated subgroups of finitely generated groups need not have finite index. We are very grateful to the referee for suggesting this coda to our paper.

References

[1] A.V. Anisimov and L.D. Seifert, Zur algebraischen characteristik dur durch kontextfreie Sprachen definierten Gruppen. Elektron. Informationsverab. Kybernet 11 (1975) 695-702.
[2] G. Baumslag, *Topics in Combinatorial Group Theory.* Lectures in Mathematics, ETH Zurich, Birkhauser Verlag (1993).
[3] R. Baer, Zur Einführung des Scharbegriffs. J. Reine Angew. Math. 160 (1929) 199-207.
[4] J. Berstel, *Transductions and Context-free Languages.* Version of December 14th 2009, online at www.infop7.org/file/659/LivreTransductions14dec2009.pdf .
[5] J. Certaine, The ternary operation $(abc) = ab^{-1}c$ of a group. Bull. Amer. Math. Soc. 49 no. 12 (1943) 869-877.
[6] C. Frougny, J. Sakarovitch and P. Schupp, Finiteness conditions on subgroups and formal language theory. Proc. London Math. Soc. (3) 58 (1989) 74-88.
[7] M. Hall Jr, Coset representations in free groups. Trans. Amer. Math. Soc. 67(2) (1949) 421-432.
[8] J.M. Howie, *Fundamentals of Semigroup Theory.* London Math. Soc. Monographs, Oxford University Press (1995).
[9] M.V. Lawson, *Inverse Semigroups.* World Scientific (1998).
[10] S.W. Margolis and J.C. Meakin, Free inverse monoids and graph immersions. Internat. J. Algebra Comput. 3 (1993) 79-99.
[11] W.D. Munn, Free inverse semigroups. Proc. London Math. Soc. 30 (1974) 385-404.
[12] J. Myhill, Finite automata and the representation of events. Wright Air Development Command Technical Report 57-624 (1957) 112-137.
[13] A. Nerode, Linear automaton transformations. Proc. Amer. Math. Soc. 9 (1958) 541-544.
[14] J.-E. Pin, *Mathematical Foundations of Automata Theory.* Version of November 30th 2016, online at www.irif.univ-paris-diderot.fr/∼jep/PDF/MPRI/MPRI.pdf .
[15] C. Reutenauer, Une topologie du monoïde libre. Semigroup Forum 18 (1979) 33-50.
[16] B. Schein, Representations of generalized groups. Izv. Vyssh. Uchebn. Zaved. Mat. 3 (1962) 164-176.
[17] B. Schein, Cosets in groups and semigroups. In *Semigroups with applications, (eds J. M. Howie, W. D. Munn, and H. J. Weinert),* World Scientific, Singapore (1992) 205-221.
[18] M. Sipser, *Introduction to the Theory of Computation.* PWS Publishing Company (1997).

Mathematics Department, Princess Nourah bint Abdulrahman University, BOX: 84428 Riyadh, Kingdom of Saudi Arabia
E-mail address: asalali@pnu.edu.sa

School of Mathematical and Computer Sciences and the Maxwell Institute for the Mathematical Sciences, Heriot-Watt University, Edinburgh EH14 4AS, U.K.
E-mail address: N.D.Gilbert@hw.ac.uk